

Impedances of Warm Plasmas near the Lower Hybrid Resonance

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With full electromagnetic treatment electron pressure effects are included in the calculation of plasma impedances and power transfer to a plasma cylinder in the domain of the lower hybrid resonance. The general results are applied to cases of interest in radio frequency discharges with azimuthally symmetric excitation and zero coil-plasma-gap. For these specific situations the effects of electron pressure are found to be small.

I. Introduction

The problem of power transfer to a plasma cylinder from a surrounding radio frequency coil has led to great interest in the frequency range of the lower hybrid frequency at the geometrical mean of ion and electron cyclotron frequencies, where resonant power transfer may be expected. KÖRPER¹ has given a theoretical treatment of this problem on the basis of a cold homogeneous plasma with collisions and a coil idealized as current sheet of rotational symmetry, assuming both plasma and coil to be of infinite axial extent. The complication of a gap between plasma and radio frequency coil can easily be accounted for².

In view of the great influence of the angle between the direction of wave propagation and static magnetic field on the resonance frequency in simple plane plasma models and in view of experimental indications of the influence of reduced lengths of the plasma column and of increased ion masses^{3,4}, recently the assumption of infinite axial extent has been dropped and an analysis for both finite plasma column and coupling coil has been carried through⁵. This work has pointed out the possibility of drastic modifications by the inclusion of finite length effects (finite $k_{||}$) for specific situations, i.e. for (axially) short plasma columns with heavy ion masses. However, for many conditions, such as considered in recent experiments with hydrogen plasmas of relatively large axial extent⁶, the finite length effects have been shown

to be of minor importance. In particular the strong role of radial modes in the vicinity of the lower hybrid frequency ("geometrical coupling resonances"⁵) is not appreciably altered due to finite $k_{||}$ in these situations. The following studies are mainly addressed to similar sets of conditions (ratios of electron to ion temperature in the order of 10^2).

The studies above are limited to cold plasma approaches — including collisions — and neglect thermal effects. DEMIDOV et al.⁷ have pointed out the potential importance of these effects near the lower hybrid frequency. This study has considered these effects in the fluid approximation including the electron pressure; a comparison to a kinetic treatment, retaining (electron) Larmor radius terms up to fourth order, has yielded essentially identical results. In this investigation the influence of ion pressure has also been discussed and regarded as being of secondary importance in the frequency range of interest. The cited work has been concerned, however, with plasmas of infinite extent and thus confined to analytical studies of dispersion relations. The same holds true for the calculations of OAKES and SCHLÜTER⁸.

The boundary value problem to be solved in case of a bounded plasma (for instance, in case of a plasma cylinder of finite radius) can introduce important modifications of the resonant behavior of a plasma. This is well known for cold plasma treatments^{1,5}. The characteristics of bounded plasmas can in turn be modified by inclusion of the

¹ K. KÖRPER, Z. Naturforsch. **129**, 814 [1957]; **159**, 220, 226, 235 [1960].

² M. E. OAKES, H. SCHLÜTER, and R. SKIPPING, Phys. Letters **25A**, 482, 704 [1967].

³ H. SCHLÜTER and C. J. RANSOM, Ann. Phys. (N.Y.) **33**, 360 (1965).

⁴ G. S. MILLS, M. E. OAKES, and H. SCHLÜTER, Phys. Letters **21**, 45 [1966].

⁵ C. R. SKIPPING, M. E. OAKES, and H. SCHLÜTER, Phys. Fluids **12**, 1886 [1969].

⁶ D. E. HASTI, M. E. OAKES, and H. SCHLÜTER, Plasma Phys. **4**, 787 [1970].

⁷ V. D. DEMIDOV, D. A. FRANK-KAMENETSKII, and V. L. YAKIMENKO, Soviet Phys.—Tech. Phys. **7**, 875 [1963].

⁸ M. E. OAKES and H. SCHLÜTER, Ann. Phys. (N.Y.) **35**, 396 [1965].



electron pressure which influences the cold plasma mode and introduces an additional mode simultaneously. This problem is studied in the present investigation with full electromagnetic treatment which encompasses and stresses the case of azimuthally symmetric excitation excluded by quasi-static treatments⁹.

II. Derivation of General Relations

Assuming perturbations $\exp(-i\omega t)$ of homogeneous plasmas the following linearized momentum transfer equations are taken as basis (with the usual notations):

$$-i\omega_j \mathbf{v}_j = \varepsilon_j \frac{e}{m_j} \mathbf{E} + \varepsilon_j \Omega_j \mathbf{v}_j \times \hat{\mathbf{e}}_z - \frac{U_j^2}{N_j} \nabla n_j. \quad (1)$$

The stationary magnetic field is taken to point in the direction of the axial unit vector $\hat{\mathbf{e}}_z$. The subscript j denotes the electron (e) and — here single — ion (i) component. These abbreviations are used:

$$\begin{aligned} & \lambda_e^2 \lambda_i^2 \nabla_\perp^6 H_z + [\lambda_e^2(1 - A_i) + \lambda_i^2(1 - A_e) + k_0^2 \lambda_e^2 \lambda_i^2(1 - A_e - A_i + \beta_e^2 A_e + \beta_i^2 A_i)] \nabla_\perp^4 H_z \\ & + [1 - A_e - A_i + k_0^2 \lambda_e^2 \{(1 - A_i)(1 - A_e - A_i + \beta_e^2 A_e) - \beta_i^2 A_i^2\} \\ & + k_0^2 \lambda_i^2 \{(1 - A_e)(1 - A_e - A_i + \beta_i^2 A_i) - \beta_e^2 A_e^2\}] \nabla_\perp^2 H_z \\ & + k_0^2 [(1 - A_e - A_i)^2 - (\beta_i A_i - \beta_e A_e)^2] H_z = 0. \end{aligned} \quad (5)$$

In (5) $E_z = 0$ is assumed, since in the following no axial currents of an exciting coil shall be considered. As pointed out in the introduction, finite length effects are of minor interest here; hence $k_\parallel = 0$, $\mathbf{k} = \mathbf{k}_\perp$, and z -dependencies have been neglected. Various abbreviations used in (5) are defined as follows:

$$\begin{aligned} A_j &= \frac{\omega_{pj}^2 / \omega \omega_j}{1 - \beta_j^2}, \quad \omega_{pj}^2 = e^2 N_j / \varepsilon_0 m_j, \\ \lambda_j^2 &= \frac{U_j^2 / \omega \omega_j}{1 - \beta_j^2}, \quad \beta_j = \Omega_j / \omega_j, \quad k_0^2 = \varepsilon_0 \mu_0 \omega^2. \end{aligned} \quad (6)$$

Equation (5) gives the dispersion relation for an infinite homogeneous plasma if one replaces ∇_\perp by $i\mathbf{k}_\perp$. In the subsequent discussion only the electron-pressure shall be considered, the ion-pressure being neglected. Some modifying effects of finite ion-pressure (in particular when it becomes comparable to the electron pressure) have recently been discussed in a quasi-static treatment⁹.

⁹ P. E. VANDENPLAS, A. M. MESSIAEN, J.-L. MONFORT, and J. J. PAPIER, *Plasma Phys.* **12**, 391 [1970].

$$\begin{aligned} \omega_j &= \omega(1 + i\nu_j/\omega), \\ U_j^2 &= \gamma_j K T_j / m_j, \\ \Omega_j &= |e| B_0 / m_j. \end{aligned} \quad (2)$$

The collision frequencies ν_j accounts for collisions of component j with neutrals (n) only, since the following analysis is mainly addressed to cases of weakly ionized plasmas. However, the simplified treatment of the collision term in (1) can be extended to account also for collision between ions and electrons and perturbations of the neutral fluid^{5,8}.

Combination of (1) with the continuity equations

$$-i\omega n_j + N_j \nabla \cdot \mathbf{v}_j = 0 \quad (3)$$

and the Maxwellian equations

$$\begin{aligned} \nabla \times \mathbf{E} &= i\omega \mu_0 \mathbf{H}, \\ \nabla \times \mathbf{H} &= -i\omega \varepsilon_0 \mathbf{E} + e \sum_j \varepsilon_j N_j \mathbf{v}_j, \\ \varepsilon_0 \nabla \cdot \mathbf{E} &= e \sum_j \varepsilon_j n_j, \quad \nabla \cdot \mathbf{H} = 0, \end{aligned} \quad (4)$$

leads to

Now the general relation (5) reduces to

$$(a \nabla_\perp^4 + b \nabla_\perp^2 + c) H_z = 0, \quad (7)$$

with

$$\begin{aligned} a &= U_e^2 A_e (1 - A_i) / \omega_{pe}^2, \\ b &= b_0 + k_0^2 U_e^2 b_i / \omega_{pe}^2, \\ c &= k_0^2 [b_0^2 - (\beta_i A_i - \beta_e A_e)^2], \end{aligned} \quad (8)$$

and

$$\begin{aligned} b_0 &= 1 - A_e - A_i, \\ b_i &= A_e [(1 - A_i)(1 - A_e - A_i) + A_e(1 - A_i) \beta_e^2 \\ & \quad - A_i^2 \beta_i^2]. \end{aligned}$$

Equation (7) implies the cold collisional plasma dispersion relation when $U_e^2 = 0$, i.e. for zero electron pressure. Setting $k_0^2 = 0$ will lead to the quasi-static approximation if correspondingly the H_z is also replaced by the electrostatic potential⁹. Related dispersion relations have been discussed by many authors, for instance by DEMIDOV et al.⁷ and OAKES and SCHLÜTER⁸. Neglecting the ion pressure has lead to an asymmetry with respect to ion- and electron-parameters in (7). Equation (7)

can be rearranged to the form

$$(\nabla_{\perp}^2 + k_1^2)(\nabla_{\perp}^2 + k_2^2)H_z = 0, \quad (9)$$

where k_1^2 and k_2^2 can be identified to be the propagation constants solving the dispersion relation of infinite plane plasma

$$a k_{\perp}^4 - b k_{\perp}^2 + c = 0. \quad (10)$$

The labelling of the two roots of k_{\perp}^2 is arbitrary in principle. For instance in Ref. ⁸ the labelling is chosen to yield always curves without discontinuities near the lower hybrid frequency ω_0 (i.e. $(\Omega_i \Omega_e)^{1/2}$ for $\omega_{pe}^2/\Omega_e^2 \gg 1$). However, for the following the labelling is chosen so that k_2^2 denotes the root which asymptotically (both for $\omega^2 \ll \omega_0^2$ and $\omega^2 \gg \omega_0^2$) approaches the value $c/b_0 = k_c^2$, which is the sole well known solution for a cold plasma treatment neglecting the electron pressure. k_c^2 has a singularity at ω_0 in the absence of collisions; inclusion of collisions removes this singularity of k_c^2 in a well studied fashion¹. The root k_2^2 solving (10) — with inclusion of the electron pressure — does not exhibit the above singularity of k_c^2 for zero collisions, but shows a finite discontinuity; introduction of collisions does not remove this discontinuity of k_2^2 , unless the collisions are

numerous enough to satisfy a certain condition which will be met later on. This condition has already been studied previously^{7,8}. The additional root k_1^2 introduced by electron pressure shows a discontinuity near ω_0 complementary to that of k_2^2 . When the condition of numerous collisions just cited is fulfilled, k_1^2 acquires a usual cut-off behavior near ω_0 whereas k_2^2 approaches k_c^2 even at $\omega \approx \omega_0$. Plots of both roots of a dispersion relation with finite electron pressure have previously been given for different situations⁸, in particular for experimental conditions which are primarily referred to here^{3,6}.

Now solutions of (9) shall be obtained with an assumed azimuthal periodicity of type $\exp(in\theta)$. Inside the plasma the solution for H_z is

$$H_z^{(p)} = [A_n J_n(k_1 r) + B_n J_n(k_2 r)] e^{in\theta}. \quad (11)$$

Whatever sign of the solutions of k_1^2 and k_2^2 are chosen above, the constants A_n and B_n can be thought to take into account the additional terms resulting from the inclusion of the opposite signs for k_1 and k_2 . For Poynting vector considerations below, also the solutions of \mathbf{E} -field components are needed. Combination of (1), (3), (4), and (9) leads to

$$E_r^{(p)} = K_{re} \frac{\partial}{\partial r} (\mathcal{L}_e H_z^{(p)}) + K_{\theta e} \frac{i}{r} \frac{\partial}{\partial \theta} (\mathcal{L}_e H_z^{(p)}) + \frac{i \bar{K}_{\theta}}{\epsilon_0 \omega} \frac{\partial H_z^{(p)}}{\partial r} + \frac{\bar{K}_r}{\epsilon_0 \omega} \frac{i}{r} \frac{\partial H_z^{(p)}}{\partial \theta}, \quad (12)$$

$$\text{and} \quad i E_{\theta}^{(p)} = K_{\theta e} \frac{\partial}{\partial r} (\mathcal{L}_e H_z^{(p)}) + K_{re} \frac{i}{r} \frac{\partial}{\partial \theta} (\mathcal{L}_e H_z^{(p)}) + \frac{\bar{K}_r}{\epsilon_0 \omega} \frac{\partial H_z^{(p)}}{\partial r} + \frac{i \bar{K}_{\theta}}{\epsilon_0 \omega} \frac{i}{r} \frac{\partial H_z^{(p)}}{\partial \theta}, \quad (13)$$

where

$$\begin{aligned} K_{re} &= \bar{K}_r - \beta_e i \bar{K}_{\theta}, & K_{\theta e} &= i \bar{K}_{\theta} - \beta_e \bar{K}_r, \\ \bar{K}_r &= K_r / (K_r^2 + K_{\theta}^2), & \bar{K}_{\theta} &= K_{\theta} / (K_r^2 + K_{\theta}^2), \\ K_r &= 1 - A_e - A_1, & i K_{\theta} &= \sum_j \epsilon_j \beta_j A_j, \end{aligned} \quad (14)$$

$$\mathcal{L}_e = \frac{U_e^2}{\epsilon_0 \omega \alpha \omega_{pe}^2} [k_0^2 b_1 + A_e (1 - A_1) \nabla_{\perp}^2], \quad \alpha = A_i (\beta_e + \beta_i) - \beta_e.$$

In general the operator \mathcal{L}_e is non-commutable with other differential operators of the cylindrical problem at hand. Equations (9) and (11) can be used to evaluate the components of \mathbf{E} -field completely. The constants A_n and B_n are to be evaluated with help of the boundary conditions.

A plasma-vacuum system with perfectly conducting coil of radius s coaxially surrounding the plasma of radius p is considered. The vacuum region between the coil and plasma $p \leq r \leq s$ is denoted by superscript V and the one outside the coil is denoted by W. In vacuum, solutions of the

wave equation are

$$H_z^{(V)} = [C_n J_n(k_0 r) + D_n H_n(k_0 r)] e^{in\theta}, \quad (15)$$

$$H_z^{(W)} = [G_n H_n(k_0 r)] e^{in\theta}. \quad (16)$$

The constants C_n , D_n , and G_n are determined below. The components of the electric field in vacuum can be easily determined by making use of Maxwell's equations

$$E_r^{(V,W)} = \frac{1}{\epsilon_0 \omega} \frac{i}{r} \frac{\partial H_z^{(V,W)}}{\partial \theta}, \quad (17)$$

$$i E_{\theta}^{(V,W)} = \frac{1}{\epsilon_0 \omega} \frac{\partial H_z^{(V,W)}}{\partial r}. \quad (18)$$

For the boundary conditions, an expression for the radial component of the electron velocity is called for:

$$\frac{i e N_e}{\epsilon_0 \omega} (\hat{e}_r \cdot \mathbf{v}_e) = A_e E_r^{(p)} - i \beta_e A_e E_\theta^{(p)} + \frac{\partial}{\partial r} (\mathcal{L}_e H_z^{(p)}) - \beta_e \frac{i}{r} \frac{\partial}{\partial \theta} (\mathcal{L}_e H_z^{(p)}). \quad (19)$$

The following set of usual electromagnetic and kinematic boundary conditions is applied:

$$r = p: \quad E_\theta^{(p)} = E_\theta^{(v)}, \quad (20)$$

$$H_z^{(p)} = H_z^{(v)}, \quad (21)$$

$$r = s: \quad E_\theta^{(v)} = E_\theta^{(w)}, \quad (22)$$

$$H_z^{(v)} - H_z^{(w)} = J_\theta^*, \quad (23)$$

$$r = p: \quad \hat{e}_r \cdot \mathbf{v}_e = 0, \quad (24)$$

with \hat{e}_r being the unit radial vector pointing towards the outside of the plasma. These five boundary conditions are sufficient to determine the five unknown constants A_n , B_n , C_n , D_n , and G_n . Incidentally, in case of non zero ion pressure an additional unknown constant would have been present in (11) calling for an additional kinematic boundary condition; usually invoking a zero radial component of \mathbf{v}_i at $r = p$. Above a current sheet of magnitude J_θ^* on the coil with infinite conductivity (SKIPPING et al. 5) is introduced. Equations (20)–(23) can be solved to obtain

$$G_n = -J^* \frac{\{I_2 II_1 - k_0 p I_1 J_n(k_1 p)\} + \frac{B_n}{A_n} \{I_2 II_2 - k_0 p I_1 J_n(k_2 p)\}}{\{II_1 H_n(k_0 p - k_0 p) H'_n(k_0 p) J_n(k_1 p)\} + \frac{B_n}{A_n} \{II_2 H_n(k_0 p) - k_0 p H'_n(k_0 p) J_n(k_2 p)\}}, \quad (25)$$

where B_n/A_n can be explicitly obtained from the remaining boundary condition (24):

$$B_n/A_n = -III_1/III_2. \quad (26)$$

The following definitions have been used:

$$I_1 = J'_n(k_0 s) H'_n(k_0 p) - J'_n(k_0 p) H'_n(k_0 s), \quad (27)$$

$$I_2 = J'_n(k_0 s) H_n(k_0 p) - J_n(k_0 p) H'_n(k_0 s), \quad (28)$$

$$II_t = k_t p J'_n(k_t p) [K_{\theta e}(k_0^2 \lambda_a^2 - k_t^2 \lambda_b^2) + \bar{K}_r] - n J_n(k_t p) [K_{re}(k_0^2 \lambda_a^2 - k_t^2 \lambda_b^2) + i \bar{K}_\theta], \quad (29)$$

$$III_t = k_t p J'_n(k_t p) \left[(K_{re} - \beta_e K_{\theta e} + \frac{1}{A_e})(k_0^2 \lambda_a^2 - k_t^2 \lambda_b^2) + K_{\theta e} \right] - n J_n(k_t p) \left[(K_{\theta e} - \beta_e K_{re} - \frac{\beta_e}{A_e})(k_0^2 \lambda_a^2 - k_t^2 \lambda_b^2) + K_{re} \right]; \quad t = 1, 2, \quad (30)$$

$$\lambda_a^2 = \frac{U_e^2}{\alpha \omega_{pe}^2} b_1, \quad \lambda_b^2 = \frac{U_e^2}{\alpha \omega_{pe}^2} A_e (1 - A_i) \quad (31)$$

$$\text{and} \quad J^* = \pm \frac{J_\theta^* e^{-in\theta}}{i(2/\pi k_0 s)}. \quad (32)$$

In Eq. (32) the $-$ or $+$ signs are consequences of the Wronskian relation for the functions $J_n(k_0 s)$ and $H_n(k_0 s)$. If $H_n^{(1)}(k_0 s)$ is chosen then the $-$ sign is to be taken and for $H_n^{(2)}(k_0 s)$ the $+$ sign is appropriate. The quantities I_1 and I_2 also exhibit the same characteristics. However, it can be ascertained that irrespective of the choice of Hankel functions of type 1 or 2 the final results will be the same.

The power transferred to the inside of the coil towards the plasma is obtained by integrating the Poynting vector (the real part of which represents

the real power loss time averaged)

$$\mathbf{S} = \frac{1}{2} \mathbf{E} \times \mathbf{H}^* \quad (33)$$

on the (inner) surface of the coil, i.e.

$$P_1 = - \int_A \mathbf{S}^{(v)} \cdot \hat{e}_r dA = - \frac{1}{2} \left[\int_A \{E_\theta^{(v)} H_z^{(v)*}\} s d\theta dz \right]_{r=s}. \quad (34)$$

The power radiated to infinity by the outer surface area of the coil is similarly obtained; this area is the same as above, since the usual model of an infinitely thin current sheet and coil has been introduced:

$$P_0 = \int_{A'} \mathbf{S}^{(w)} \cdot \hat{e}_r dA' = \frac{1}{2} \left[\int_{A'} \{E_\theta^{(w)} H_z^{(w)*}\} s' d\theta dz \right]_{r=s'}. \quad (35)$$

Thus the total power absorbed by the vacuum plasma system (the coil itself is assumed loss-less) is

$$P_t = P_i + P_0 = -\pi s \left[\int_z E_\theta^{(W)} J_\theta^{*\dagger} dz \right]_{r=s}. \quad (36)$$

Assuming a coil of length l having m turns per unit length in the z -direction such that

$$J_\theta^* = m I_\theta, \quad -\frac{l}{2} \leq z \leq \frac{l}{2} \quad (37)$$

one can obtain the impedance of the vacuum-plasma-system

$$Z_t = P_t / \frac{1}{2} |I_\theta|^2. \quad (38)$$

Explicitly, with (37), (36), (32), (25), (18), and (16) this is

$$Z_t = \pm \frac{(\pi k_0 m s)^2 l}{\epsilon_0 \omega} \left\{ \frac{\{I_2 \Pi_1 - I_1 k_0 p J_n(k_1 p)\} + \frac{B_n}{A_n} \{I_2 \Pi_2 - k_0 p I_1 J_n(k_2 p)\}}{\left\{ \Pi_1 \frac{H_n(k_0 p)}{H'_n(k_0 s)} - k_0 p \frac{H'_n(k_0 p)}{H'_n(k_0 s)} J_n(k_1 p) \right\} + \frac{B_n}{A_n} \left\{ \Pi_2 \frac{H_n(k_0 p)}{H'_n(k_0 s)} - k_0 p \frac{H'_n(k_0 p)}{H'_n(k_0 s)} J_n(k_2 p) \right\}} \right\}, \quad (39)$$

where all other constants are defined earlier. Prime on Bessel functions refer to the derivative with respect to the argument, and \pm again refers to the type of Hankel functions chosen as discussed earlier. An expression for the impedance due to the power transfer exclusively to the plasma can, of course, be obtained in an analogous manner from the expressions given before.

The relations presented aim mainly at the case of rotationally symmetric wave excitation, though they are formally valid for $n \neq 0$ as well. The boundary conditions for the asymmetric wave excitation in experimental situations are generally involved; however, the model above with $\exp(in\theta)$ dependencies throughout should facilitate useful estimates of the behavior in such cases.

III. Simplified Expressions

The results of the preceding chapter can be appreciably simplified without undue loss of generality by the assumption of zero gap between coil and plasma ($p = s$). Shrinking of the vacuum region V such that the coil is on the surface of the plasma, effects these reductions used from now on:

$$I_1 \equiv 0, \quad I_2 \equiv \mp i(2/\pi k_0 p), \quad I_2 J^* \equiv J_\theta^*. \quad (40)$$

Moreover, symmetric wave excitation is of prime interest here, so that in the azimuthal dependencies $e^{in\theta}$ now n is specified to be zero. The resultant expression for the total plasma-vacuum impedance Z_t can be expressed in this form:

$$Z_t = Z_w^{-1} + Z_p^{-1}. \quad (41)$$

The impedance Z_w is connected to the power radiated to the outside of the coil (vacuum part) and given by the (usually unimportant) contribution

$$Z_w = i R_c (H_1(k_0 p)/H_0(k_0 p)) \quad (42)$$

with

$$R_c = (2\pi l m^2 k_0 p / \epsilon_0 \omega). \quad (43)$$

The impedance Z_p connected with the power flow towards the plasma can be rearranged to the form

$$Z_p^{-1} = Z_1^{-1} + Z_2^{-1}. \quad (44)$$

The potentially oscillatory part of the impedances Z_1 and Z_2 are associated only with k_1 and k_2 , respectively, as it is evident from the detailed expressions obtained for Z_1 and Z_2 :

$$Z_1 = i R_c \frac{k_1 J_1(k_1 p)}{k_0 J_0(k_1 p)} [(\lambda_\theta^2 k_1^2 - P_\theta) - (\lambda_u^2 k_1^2 - P_u)(\lambda_\theta^2 k_2^2 - P_\theta)/(\lambda_u^2 k_2^2 - P_u)], \quad (45)$$

$$Z_2 = i R_c \frac{k_2 J_1(k_2 p)}{k_0 J_0(k_2 p)} [(\lambda_\theta^2 k_2^2 - P_\theta) - (\lambda_u^2 k_2^2 - P_u)(\lambda_\theta^2 k_1^2 - P_\theta)/(\lambda_u^2 k_1^2 - P_u)]. \quad (46)$$

These notations are used:

$$\lambda_\theta^2 = \frac{U_e^2}{\alpha \omega_{pe}^2} A_e (1 - A_i) K_{\theta e},$$

$$\lambda_u^2 = \lambda_\theta^2 \left(K_{re} - \beta_e K_{\theta e} + \frac{1}{A_e} \right) / K_{\theta e}, \quad (47)$$

$$P_\theta = \bar{K}_r + k_0^2 \lambda_\theta^2 b_1 / \{A_e (1 - A_i)\},$$

$$P_u = K_{\theta e} + k_0^2 \lambda_u^2 b_1 / \{A_e (1 - A_i)\}.$$

Incidentally the real power absorbed by the plasma, L , is represented by

$$L = \frac{1}{2} |U|^2 \operatorname{Re}(Z_p) / |Z_p|^2 \quad (48)$$

with U being the coil voltage, preferably to be considered as a given quantity in experiments. In the above notations with the coil current I_0

$$L = \frac{|I_0|^2}{2} \operatorname{Re}(Z_p) \frac{|Z_t|^2}{|Z_p|^2} \quad (49)$$

with $|Z_t|^2/|Z_p|^2$ commonly ≈ 1 .

In order to elucidate the relationship of these expressions for the impedances, to those of a cold plasma treatment neglecting the electron pressure (KÖRPER¹), they shall be studied for specific conditions. The region close to the lower hybrid frequency shall be of interest, i.e. these assumptions are made:

$$\omega_{pe}^2/\Omega_e^2 \gg 1, \quad (50)$$

$$v_e/\omega < 1, \quad v_e^2/\omega^2 \ll 1, \quad (51)$$

$$|\varepsilon| < 1, \quad \varepsilon^2 \ll 1, \quad (52)$$

$$\text{with } \varepsilon = 1 - \frac{\Omega_i \Omega_e}{\omega^2} + \frac{\Omega_e^2}{\omega_{pe}^2} - \frac{U_e^2}{c^2} \frac{\omega_{pi}^2}{\omega^2}. \quad (53)$$

Condition (50) assures that the lower hybrid frequency is given by $\omega \approx \omega_0$ and is not shifted towards Ω_i . Relation (51) excludes situations of overly strong damping; v_i is taken small compared to v_e . Moreover the condition

$$|b^2| \gg 4|ac| \quad (54)$$

for the coefficients of the dispersion relation (10) shall be fulfilled. Relation (54) is certainly satisfied when

$$v_e^2/\Omega_i \Omega_e \gg 2\gamma_e K T_e N_e/B_0^2/2\mu_0 \quad (55)$$

is valid. Relation (55) represents the condition mentioned earlier in the discussion of the dispersion relation (10). Thus with (55) the two roots of (10) can easily be separated. As mentioned before, now the root k_2 closely resembles the solution of the cold plasma treatment¹ not only below and above the lower hybrid frequency, but in the immediate neighbourhood $\omega \approx \omega_0$ as well. As a consequence of the above assumptions the expressions for the plasma impedance Z_p are simplified. The branch Z_2 can be rewritten as

$$Z_2 = Z_{\text{coll}} + Z_{\text{corr}} \quad (56)$$

with

$$Z_{\text{coll}} = -i R_c \frac{k_0 J_1(k_2 p)}{k_2 J_0(k_2 p)}, \quad (57)$$

$$Z_{\text{corr}} = i R_c \frac{k_2^2}{k_1^2} \frac{k_0 J_1(k_2 p)}{k_2 J_0(k_2 p)}. \quad (58)$$

Z_{coll} stands for the solution given by the treatment of KÖRPER¹ which includes collisions, but neglects the electron pressure. For the conditions considered in (50) – (54), $|Z_2| \approx |Z_{\text{coll}}|$. When the large argument asymptotic expressions are used for the Bessel functions involved and when the above assumptions are valid, in particular when (55) is well fulfilled, $|Z_1|$ becomes relatively large and unimportant and $\operatorname{Re}(Z_2) \approx \operatorname{Re}(Z_{\text{coll}})$. Further in the expressions (48) or (49) for the total power dissipated in the plasma, Z_p can be approximated by the result of the cold plasma treatment impedance Z_{coll} .

In the case of small argument developments the situation is more complicated and statements as above should not be made. In many practical situations, at least one of the two roots for k_\perp is large enough so that — if not the asymptotic range — the range of quasi-periodic behavior of Bessel functions is reached. Then numerical evaluations have to be resorted to anyhow for closer examination.

IV. Numerical Results

In a cold plasma treatment¹ the minima of $J_1(kp)$ in expression (48) (or of $J_0(kp)$ in expression (49)) are the source of potentially strong geometrical coupling resonances connected with radial modes occurring for $|k_r| > |k_i|$ on the low frequency side of the lower hybrid frequency (provided, of course, that $k \cdot p$ is large enough). Previously^{2,5} it has been stressed that these resonances are the dominant features near the lower hybrid frequency — in particular for conditions of recent experimental interest^{3,6} (even when the modifications due to finite k_\parallel are accounted for). Therefore, conditions relevant to these discussions and experimental studies are selected for numerical evaluations on the basis of (45) and (46).

In Figs. 1 and 2 situations of relatively high and low hydrogen gas pressure (and corresponding v_e), respectively, are considered in the range of interest with: $N_e = 2 \cdot 10^{12} \text{ cm}^{-3}$, $T_e = 10^5 \text{ °K}$, $\omega/2\pi = 29.6 \text{ MHz}$, $p = 3 \text{ cm}$, coil width 5 cm, single turn. Here γ_e is taken to be 1. The values of the loading resistance $|Z_p|^2/\operatorname{Re}(Z_p)$, inversely proportional to the power dissipated in the plasma, as given by (48), are plotted versus $(\Omega_i \Omega_e)^{1/2}/\omega$ in the domain of the lower hybrid frequency; the dotted curves

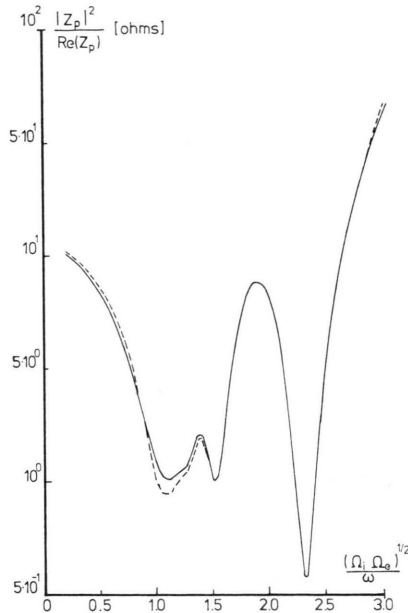


Fig. 1. Loading resistance $|Z_p|^2/\text{Re}(Z_p)$ (log-scale) as function of $(\Omega_i \Omega_e)^{1/2}/\omega \sim B_0$, for hydrogen; $\nu_e = 4.9 \cdot 10^7 \text{ sec}^{-1}$, $\nu_i = 1.13 \cdot 10^6 \text{ sec}^{-1}$.

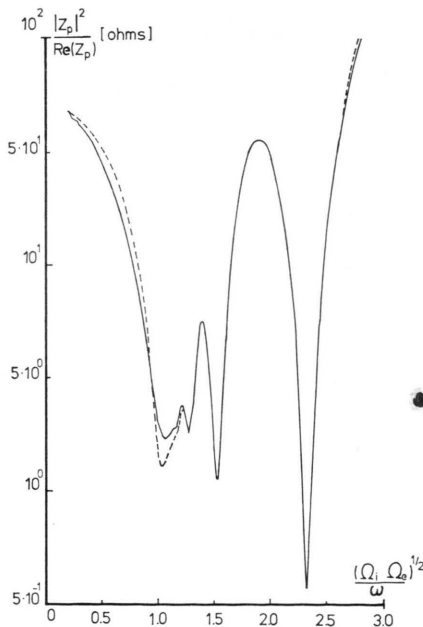


Fig. 2. Loading resistance $|Z_p|^2/\text{Re}(Z_p)$ (log-scale) as function of $(\Omega_i \Omega_e)^{1/2}/\omega \sim B_0$, for hydrogen; $\nu_e = 2.2 \cdot 10^7 \text{ sec}^{-1}$, $\nu_i = 5.06 \cdot 10^5 \text{ sec}^{-1}$.

represent the corresponding results of the cold collisional plasma treatment without the inclusion of pressure effects¹. The strongest dip in both figures is connected with the first radial mode

excitation; in Fig. 1 the effect of two (in Fig. 2 of three) such modes can clearly be seen. The figures demonstrate that the inclusion of these effects causes small quantitative differences; in particular the dominant role of the geometric coupling resonances associated with radial modes for $\Omega_i \Omega_e / \omega^2 \gtrsim 1$ has been retained. Additional calculations have shown this to be true for higher collision frequencies as well. The dominant role of these geometrical resonances remains essentially unchanged by inclusion of pressure effects even in cases of lower collision frequencies, for which inequality (55) is not fulfilled; now the modifications in the range $\Omega_i \Omega_e / \omega^2 \lesssim 1$ become more noticeable. As stated before, $k_{2r}p$ has to be sufficiently large to give rise to radial modes. Thus for conditions of main interest in previous investigations, approximations with $p_e = 0$ appear to be reasonable. However, this is not obvious without the comparisons above and should not be generalized to situations of different kind⁹. — The numerical evaluations have been performed with the TR 440 computer of the Ruhr University, Bochum.

V. Conclusions and Outlook

The effect of electron pressure on the resonant behavior near the lower hybrid resonance has been incorporated into calculations of impedances and power transfer, applicable to various practical situations of laboratory plasmas. The assumption of quasi-static approximation has been avoided. For the case of azimuthal symmetry, of prime interest here, simplified expressions have been presented and by inspection of the formal structure of the resultant expressions the difference and connection to a cold plasma treatment¹ have been pointed out. Numerical evaluations have ascertained the dominant role of radial modes (geometrical coupling resonances) in the range of the lower hybrid frequency (situated nearby at the low frequency side), in particular for conditions of recent experimental investigations⁶ currently being pursued further.

It should be emphasized that the limited change caused by the inclusion of electron pressure effects in the situations considered does not imply the absence of drastic changes in other ranges and for other conditions⁹, which shall be the subject of future studies based on the expressions presented.

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Thermodynamik heterogener Gasgleichgewichte

I. Freie Bildungsenthalpie ΔG_B^0 der Verbindungen in den Wolfram-Halogen-Systemen

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Zur Beurteilung der chemischen Transportreaktionen in den Wolfram-Halogen Systemen werden nach dem Verfahren der 3. Ulichischen Näherung Gleichgewichtskonstanten K_p und freie Bildungsenthalpien ΔG_B^0 der Verbindungen WF_2 , WF_4 , WF_5 , WBr_2 , WBr_4 und der hypothetischen Verbindungen WJ , WJ_2 , WJ_4 , WJ_5 , WJ_6 bis zur Temperatur des flüssigen Wolframs ermittelt.

Im Zusammenhang mit bekannten Daten zeigt sich, daß in den Systemen Wolfram-Chlor und Wolfram-Brom die Reaktionen über die Verbindungen WX_4 und WX_2 ablaufen, während im System Wolfram-Fluor daneben die Verbindungen WX_5 und WX_6 eine Rolle spielen. Die Daten des Systems Wolfram-Jod zeigen, daß die reinen Wolfram-Jod-Verbindungen energetisch nicht stabil sind. Die in der Literatur beschriebenen experimentellen Befunde von der Nichtexistenz dieser Verbindungen werden dadurch vom Standpunkt der Thermodynamik bestätigt.

Problemstellung

Die in den Wolfram-Halogenglühlampen ablaufenden Reaktionen stellen ein sehr komplexes System von chemischen Gleichgewichtsreaktionen dar, an dem die Konstituenten Wolfram – Halogen – Sauerstoff – Wasserstoff – Kohlenstoff beteiligt sind. Um einen Einblick in den Ablauf und das Ineinandergreifen der einzelnen Reaktionssysteme zu erhalten, sollen in einer Reihe von Arbeiten die Gleichgewichtszusammensetzung der Gasphase sowie die Richtung der in den Reaktionssystemen ablaufenden chemischen Transportreaktionen für die einzelnen Systeme auf der Grundlage der chemischen Thermodynamik berechnet werden.

Die Ausgangsbasis der Berechnungen bilden die thermodynamischen Daten der JANAF-Tabellen¹, die eine kritisch aufeinander abgestimmte Zusammenstellung von neuesten Literaturdaten darstellen. Gegenstand der vorliegenden Arbeit ist die Abschätzung derjenigen thermodynamischen Gleichgewichtsdaten der Systeme Wolfram – Fluor und Wolfram – Brom, die nicht in den JANAF-Tabellen enthalten sind. Gleichzeitig sollen Näherungswerte für das System Wolfram – Jod abgeschätzt werden.

Zur Ermittlung unbekannter Gleichgewichtsdaten stehen zunächst zwei Methoden zur Verfügung: Die Methode der statistischen Thermodynamik sowie die Methode der phänomenologischen Thermodynamik.

Beim Verfahren der statistischen Thermodynamik² erfordert die Berechnung der Gleichgewichtskonstanten K_p die Kenntnis einer größeren Anzahl molekularer Daten, deren Abschätzung aus Mangel an verlässlichen Vergleichsdaten recht schwierig ist. Geringfügige Fehler in den Ausgangsgrößen können sich zu stark fehlerhaften, praktisch unbrauchbaren Gleichgewichtsdaten akkumulieren.

Beim Verfahren der phänomenologischen Thermodynamik³ unter Verwendung der 3. Ulichischen Näherung

$$\ln K_p = -\frac{\Delta H_{298}^0}{RT} + \frac{\Delta S_{298}^0}{R} + \frac{a}{R} f\left(\frac{T}{298}\right)$$

werden empirische oder halbempirische Standardreaktionsenthalpien und -entropien verwendet.

Die im Korrekturglied als Faktor a enthaltene Reaktionsmolwärme läßt sich dabei meist aus gleichartigen Reaktionen mit ausreichender Genauigkeit gewinnen³.

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¹ JANAF-Thermochemical Tables and Addenda I–III, Dow Chemical Comp., Midland, Michigan 1965–68.

² J. E. MAYER u. M. GOEPPERT-MAYER, Statistical Mechanics, John Wiley & Sons, New York 1940.

³ O. KUBASCHEWSKI u. E. L. EVANS, Metallurgische Thermodynamik, VEB Verlag der Wissensch., Berlin 1959.